Unit 2 : Magnetostatics

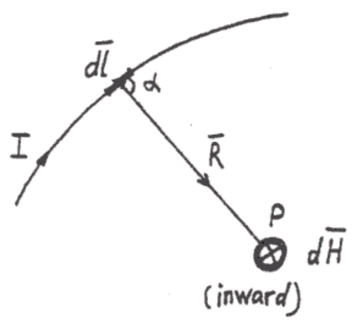
TABLE OF CONTENTS

Magnetostatics

If charges are moving with constant velocity, a static magnetic (or magnetostatic) field is produced. Thus, magnetostatic fields originate from currents (for instance, direct currents in current-carrying wires).

Most of the equations we have derived for the electric fields may be readily used to obtain corresponding equations for magnetic fields if the equivalent analogous quantities are substituted.

Biot – Savart's Law



$$dH = k\frac{Idl\sin\alpha}{R^2}$$

IN SI units, $k = \frac{1}{4\pi}$, so

$$dH = k\frac{Idl\sin\alpha}{4\pi R^2}$$

The magnetic field intensity dH produced at a point P by the differential current element Idl is proportional to the product of Idl and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

Using the definition of cross product $(\overline{A} \times \overline{B} = AB \sin\theta_{AB} \overline{e}_n)$
we can represent the previous equation in vector form as

$$d\overline{H} = \frac{I\overline{dl} \times \overline{e}_r}{4\pi R^2} = \frac{I\overline{dl} \times \overline{R}}{4\pi R^3} \qquad R = \left|\overline{R}\right| \qquad \overline{e}_r = \overline{R}/R$$

The direction of $d\overline{H}$ can be determined by the right-hand rule or by the right-handed screw rule.

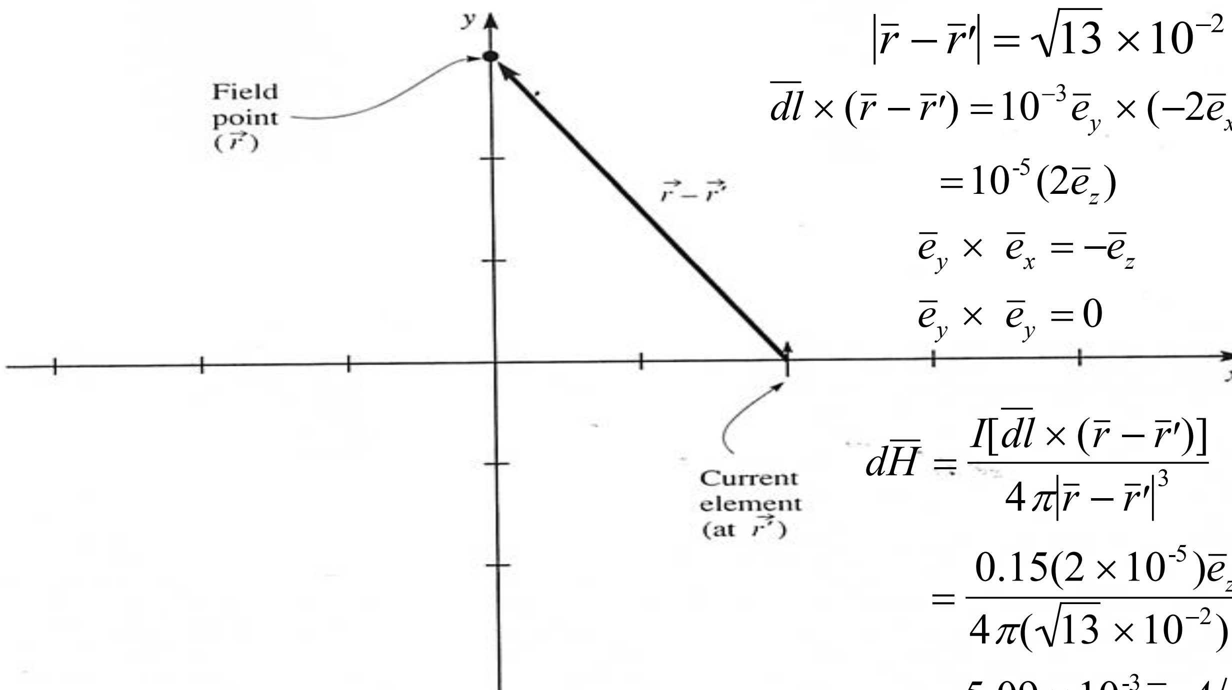
If the position of the field point is specified by $\overline{r}$ and the position of the source point by $\overline{r}'$

$$d\overline{H} = \frac{I(\overline{dl} \times \overline{e}_{r'r})}{4\pi\left|\overline{r} - \overline{r}'\right|^2} = \frac{I[\overline{dl} \times (\overline{r} - \overline{r}')]}{4\pi\left|\overline{r} - \overline{r}'\right|^3}$$

where $\overline{e}_{r'r}$ is the unit vector directed from the source point to the field point, and $\left|\overline{r} - \overline{r}'\right|$ is the distance between these two points.

<u>Example</u>. A current element is located at x=2 cm,y=0, and z=0. The current has a magnitude of 150 mA and flows in the +y direction. The length of the current element is 1mm. Find the contribution of this element to the magnetic field at x=0, y=3cm, z=0

$\overline{dl} = 10^{-3}\overline{e}_y \qquad \overline{r} = 3\times10^{-2}\overline{e}_y \quad \overline{r}' = 2*10^{-2}\overline{e}_x \qquad (\overline{r}-\overline{r}') = (-2\overline{e}_x + 3\overline{e}_y)\times10^{-2}$



$$\left|\overline{r}-\overline{r}'\right| = \sqrt{13}\times10^{-2}$$

$$\overline{dl}\times(\overline{r}-\overline{r}') = 10^{-3}\overline{e}_y \times (-2\overline{e}_x + 3\overline{e}_y)\times10^{-2}$$

$$= 10^{-5}(2\overline{e}_z)$$

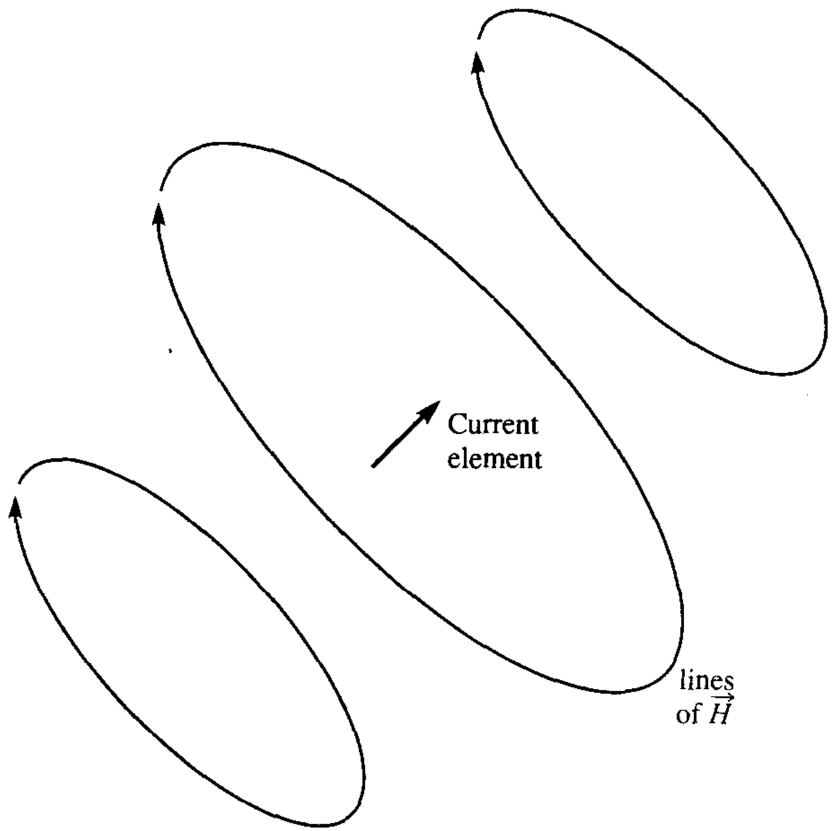
$$\overline{e}_y \times \overline{e}_x = -\overline{e}_z$$

$$\overline{e}_y \times \overline{e}_y = 0$$

$$d\overline{H} = \frac{I[\overline{dl}\times(\overline{r}-\overline{r}')]}{4\pi\left|\overline{r}-\overline{r}'\right|^3}$$

$$= \frac{0.15(2\times10^{-5})\overline{e}_z}{4\pi(\sqrt{13}\times10^{-2})^3}$$

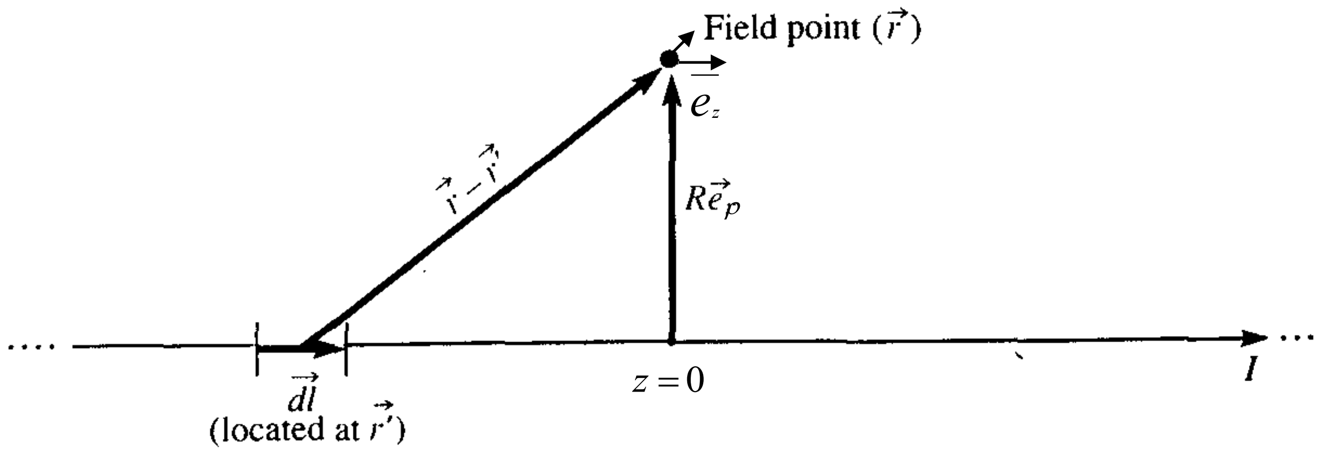
$$= 5.09\times10^{-3}\,\overline{e}_z\ A/m$$



The field produced by a wire can be found adding up the fields produced by a large number of current elements placed head to tail along the wire (the principle of superposition).

Figure 5.2 The magnetostatic field of a current element is in the form of circles around the current that produces it. The direction of the field can be found from the right-hand rule: If the thumb of the right hand points in the direction of the current, the fingers point in the direction of the field.

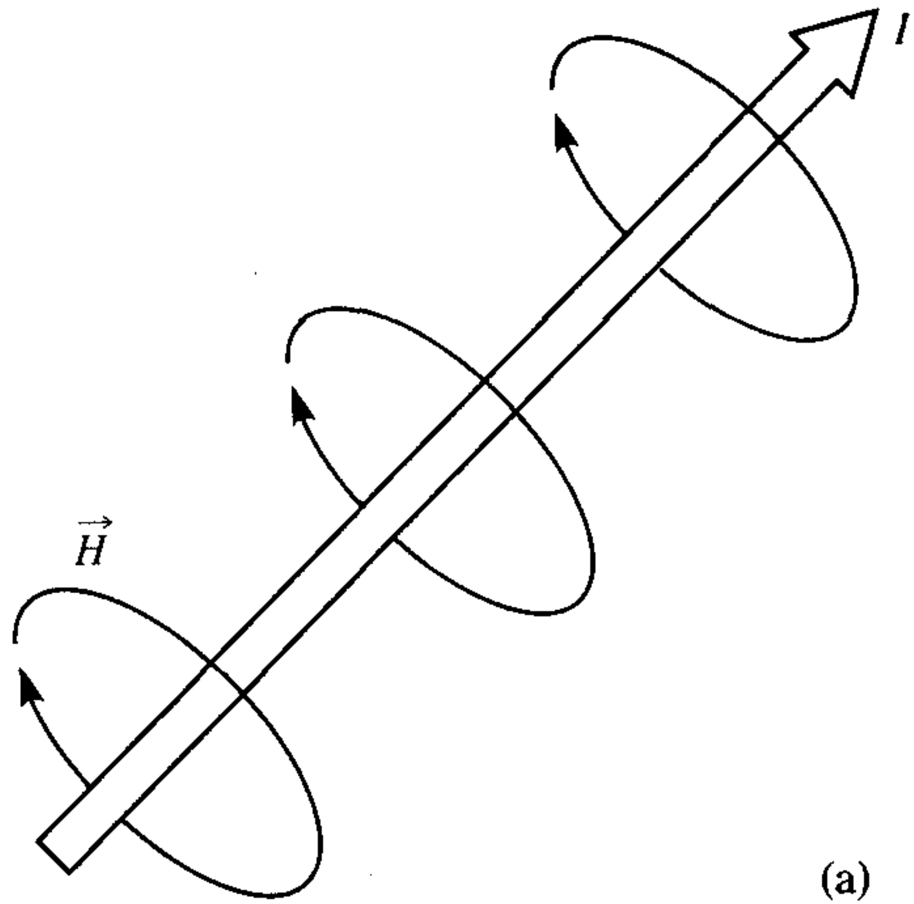
Consider an infinitely long straight wire located on the z axis and carrying a current I in the +z direction. Let a field point be located at z=0 at a radial distance R from the wire.



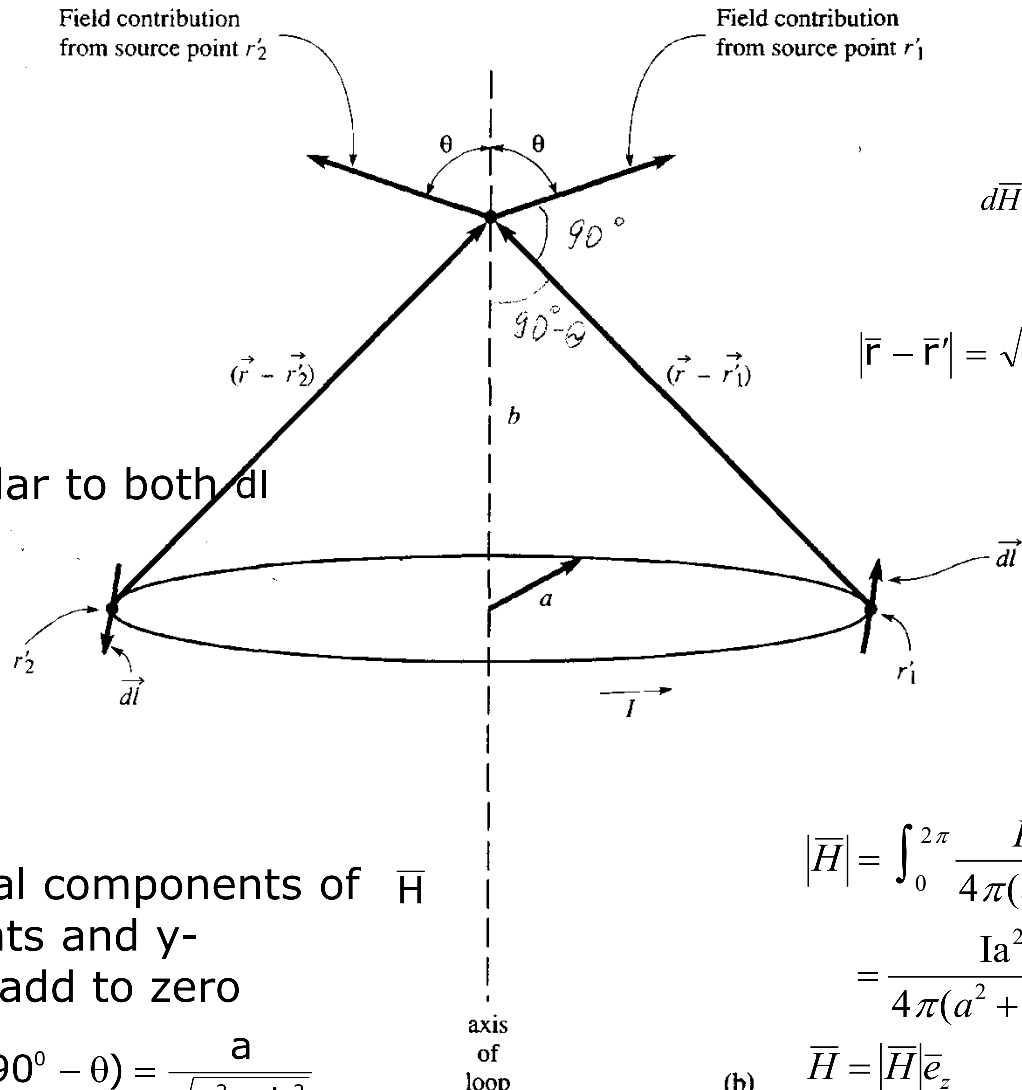
$$|\bar{r}-\bar{r}'| = (R^2+Z^2)^{1/2} \qquad d\bar{H} = \frac{I(\bar{dl}\times\bar{e}_{r'r})}{4\pi|\bar{r}-\bar{r}'|^2} = \frac{Idz(\bar{e}_z-\bar{e}_{r'r})}{4\pi|\bar{r}-\bar{r}'|^2} = \frac{Idz\ \overbrace{\sin\alpha}^{\frac{R}{(R^2+Z^2)^{1/2}}}\ \bar{e}_\phi}{4\pi(R^2+Z^2)}$$

$$\bar{H} = \int_{z=-\infty}^{\infty} \frac{Idz}{4\pi(R^2+Z^2)}\frac{R}{(R^2+Z^2)^{1/2}} = \int_{-\infty}^{\infty}\frac{1}{(R^2+Z^2)^{3/2}}dz = \frac{IR}{4\pi}\left(\frac{Z}{R^2\sqrt{R^2+Z^2}}\right)\Bigg|_{-\infty}^{\infty}$$

$$= \frac{I}{4\pi R}(1+1) = \frac{I}{2\pi R}$$



(a)



$$d\overline{H} = \frac{I(d\bar{l} \times \bar{e}_{r'r})}{4\pi|\bar{r}-\bar{r}'|^2}$$

$$|\bar{r}-\bar{r}'| = \sqrt{a^2+b^2} = \text{const}$$

$$dl = a\,d\phi$$

(perpendicular to both dl and $\bar{e}_{r'r}$)

The horizontal components of $\overline{H}$ (x-components and y-components add to zero

$$\cos\theta = \sin(90^0 - \theta) = \frac{a}{\sqrt{a^2+b^2}}$$

$$|\overline{H}| = \int_0^{2\pi} \frac{Ia\,d\phi}{4\pi(a^2+b^2)} \cdot \frac{a}{(a^2+b^2)^{1/2}}$$

$$= \frac{Ia^2}{4\pi(a^2+b^2)^{3/2}} \cdot 2\pi = \frac{Ia^2}{2(a^2+b^2)^{3/2}}$$

$$\overline{H} = |\overline{H}|\bar{e}_z$$

Magnetic Flux Density

The magnetic flux density vector is related to the magnetic field intensity $\overline{H}$ by the following equation

$$\overline{B} = \mu\overline{H}, \qquad \text{T (tesla) or Wb/m}^2$$

where μ is the permeability of the medium. Except for ferromagnetic materials (such as cobalt, nickel, and iron), most materials have values of μ very nearly equal to that for vacuum,

$$\mu_o = 4\pi \cdot 10^{-7} \qquad \text{H/m} \qquad \text{(henry per meter)}$$

The magnetic flux through a given surface S is given by

$$\Psi = \int_S \overline{B} \cdot d\overline{s}, \qquad \text{Wb (weber)}$$

Magnetic Force

One can tell if a magnetostatic field is present because it exerts forces on moving charges and on currents. If a current element $I\overline{dl}$ is located in a magnetic field $\overline{B}$, it experiences a force

$$d\overline{F} = I\overline{dl} \times \overline{B}$$

This force, acting on current element $I\overline{dl}$, is in a direction perpendicular to the current element and also perpendicular to $\overline{B}$. It is largest when $\overline{B}$ and the wire are perpendicular.

The force acting on a charge q moving through a magnetic field with velocity $\overline{v}$ is given by

$$\overline{F} = q(\overline{v} \times \overline{B})$$

The force is perpendicular to the direction of charge motion, and is also perpendicular to $\overline{B}$.

The Curl Operator

This operator acts on a vector field to produce another vector field. Let $\overline{B}(x,y,z)$ be a vector field. Then the expression for the curl of $\overline{B}$ in rectangular coordinates is

$$\nabla \times \overline{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\overline{e}_x + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\overline{e}_y + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\overline{e}_z$$

where B_x, B_y, and B_z are the rectangular components of

The curl operator can also be written in the form of a determinant:

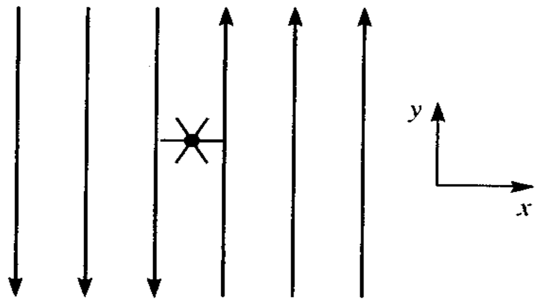
$$\nabla \times \overline{B} = \begin{vmatrix} \overline{e}_x & \overline{e}_y & \overline{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

The physical significance of the curl operator is that it describes the "rotation" or "vorticity" of the field at the point in question. It may be regarded as a measure of how much the field curls around that point.

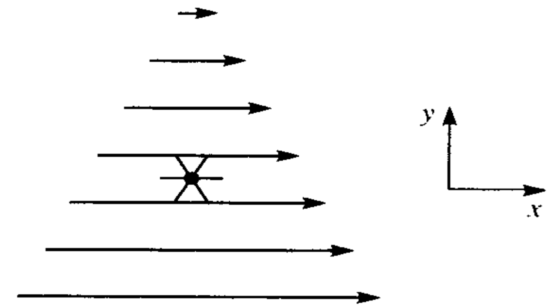
The curl of $\overline{B}$ is defined as an axial (or rotational) vector whose magnitude is the maximum circulation of $\overline{B}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

$$\text{curl}\,\overline{B} = \text{rot}\,\overline{B} = \nabla \times \overline{B} = \left(\lim_{\Delta s \to 0} \frac{\oint_L \overline{B} \cdot \overline{dl}}{\Delta s} \right)_{max} \overline{a}_n$$

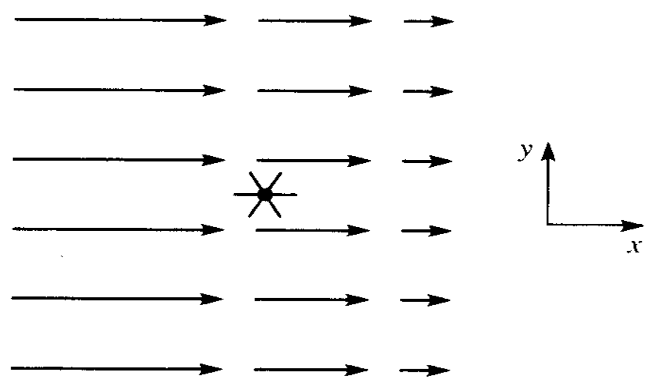
where the area Δs is bounded by the curve L, and $\overline{a}_n$ is the unit vector normal to the surface Δs and is determined using the right – hand rule.



(c)



(d)



(e)

$$\nabla \times \overline{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \overline{e}_x + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \overline{e}_y + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \overline{e}_z$$

Classification of Vector Fields

A vector field is uniquely characterized by its divergence and curl (Helmholtz's theorem). The divergence of a vector field is a measure of the strength of its flow source and the curl of the field is a measure of the strength of its vortex source.

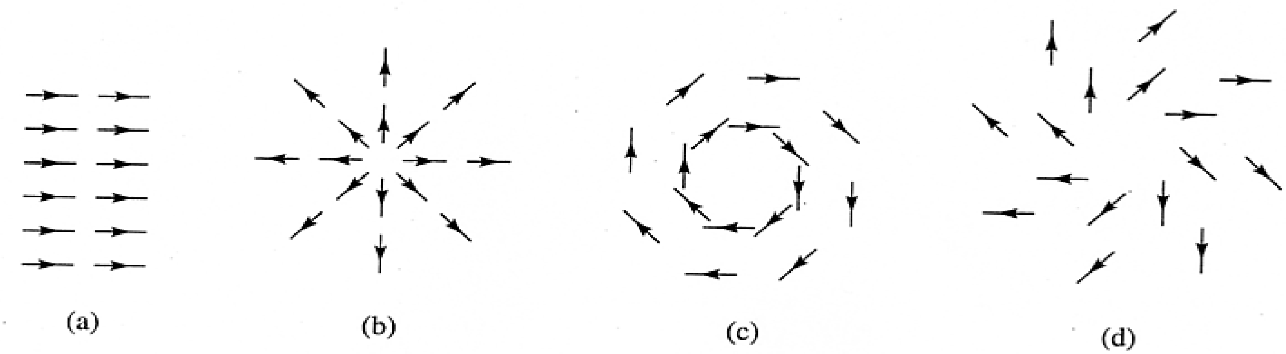


Figure 3.23 Typical fields with vanishing and nonvanishing divergence or curl.
(a) $\mathbf{A} = k\mathbf{a}_x$, $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = 0$,
(b) $\mathbf{A} = k\mathbf{r}$, $\nabla \cdot \mathbf{A} = 3k$, $\nabla \times \mathbf{A} = 0$,
(c) $\mathbf{A} = \mathbf{k} \times \mathbf{r}$, $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = 2\mathbf{k}$,
(d) $\mathbf{A} = \mathbf{k} \times \mathbf{r} + c\mathbf{r}$, $\nabla \cdot \mathbf{A} = 3c$, $\nabla \times \mathbf{A} = 2\mathbf{k}$.

If $\nabla \cdot \overline{A} = 0$ then $\overline{A}$ is said to be solenoidal or divergenceless. Such a field has neither source nor sink of flux.

Since $\nabla \cdot (\nabla \times \overline{F}) = 0$ (for any $\overline{F}$), a solenoidal field $\overline{A}$ can always be expressed in terms of another vector $\overline{F}$:

$$\overline{A} = \nabla \times \overline{F}$$

If $\nabla \times \overline{A} = 0$ then $\overline{A}$ is said to be irrotational (or potential, or conservative). The circulation of $\overline{A}$ around a closed path is identically zero.

Since $\nabla \times \left(\nabla V\right) = 0$ (for any scalar V), an irrotational field $\overline{A}$ can always be expressed in terms of a scalar field V:

$$\overline{A} = -\nabla V$$

Magnetic Vector Potential

Some electrostatic field problems can be simplified by relating the electric potential V to the electric field intensity $\overline{E}(\overline{E} = -\nabla V)$. Similarly, we can define a potential associated with the magnetostatic field $\overline{B}$:

$$\overline{B} = \nabla \times \overline{A}$$

where $\overline{A}$ is the magnetic vector potential.

Just as we defined in electrostatics

$$V(\overline{r}) = \int \frac{dq(\overline{r}')}{4\pi\varepsilon|\overline{r} - \overline{r}'|} \qquad \text{(electric scalar potential)}$$

we can define

$$\overline{A}(\overline{r}) = \int_L \frac{\mu I(\overline{r}')\overline{dl'}}{4\pi|\overline{r} - \overline{r}'|} \quad \frac{Wb}{m} \qquad \text{(for line current)}$$

The contribution to $\overline{A}$ of each differential current element $I\overline{dl'}$ points in the same direction as the current element that produces it.

The use of $\overline{A}$ provides a powerful, elegant approach to solving EM problems (it is more convenient to find $\overline{B}$ by first finding $\overline{A}$ in antenna problems).

The Magnetostatic Curl Equation

Basic equation of magnetostatics, that allows one to find the current when the magnetic field is known is

$$\nabla \times \overline{H} = \overline{J}$$

where $\overline{H}$ is the magnetic field intensity, and $\overline{J}$ is the current density (current per unit area passing through a plane perpendicular to the flow).

The magnetostatic curl equation is analogous to the electric field source equation $\nabla \cdot \overline{D} = \rho$

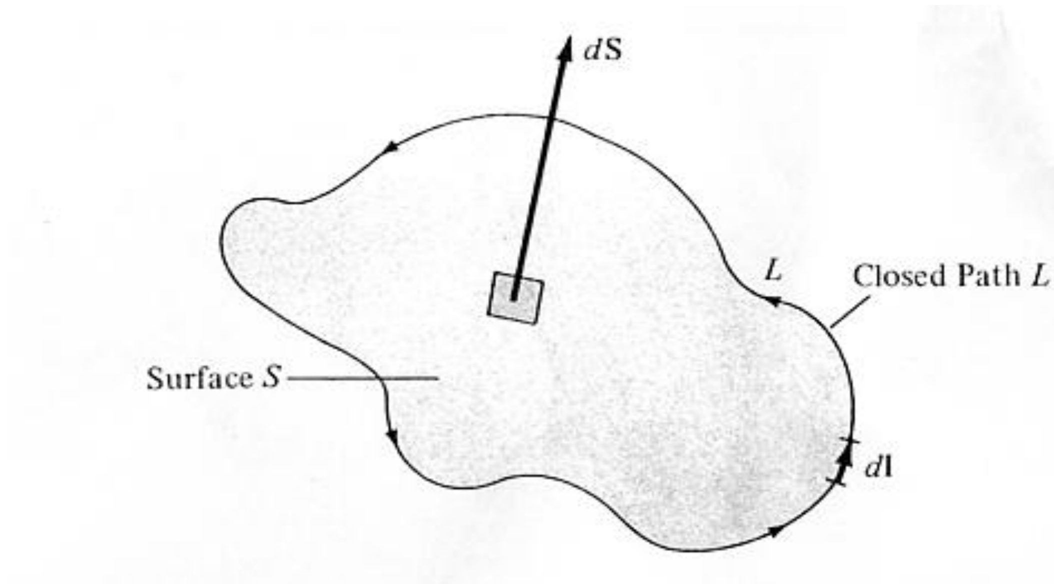
Stokes' Theorem

This theorem will be used to derive Ampere's circuital law which is similar to Gauss's law in electrostatics.

According to Stokes' theorem, the circulation of $\overline{A}$ around a closed path L is equal to the surface integral of the curl of $\overline{A}$ over the open surface S bounded by L.

$$\oint_L \overline{A} \cdot \overline{dl} = \int_S \left(\nabla \times \overline{A}\right) \cdot \overline{ds}$$

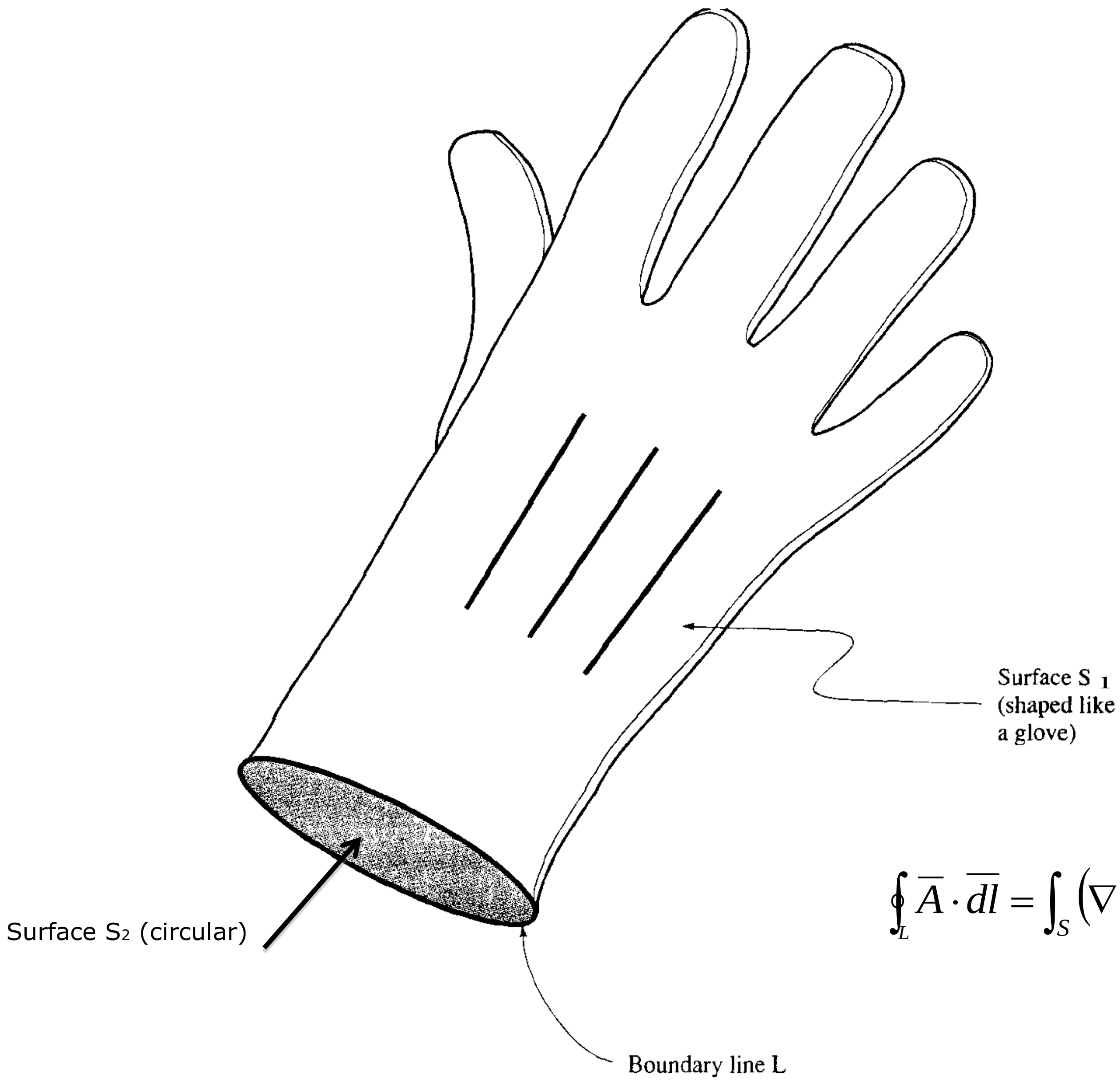
The direction of integration around L is related to the direction of $\overline{ds}$ by the right-hand rule.



Determining the sense of dl and dS involved in Stokes's theorem

Stokes' theorem converts a surface integral of the curl of a vector to a line integral of the vector, and vice versa.
(The divergence theorem relates a volume integral of the divergence of a vector to a surface integral of the vector, and vice versa).

$$\int_{v} \nabla \cdot \overline{A} dv = \int_{S} \overline{A} \cdot \overline{ds}$$



$$\oint_L \overline{A} \cdot \overline{dl} = \int_S \left(\nabla \times \overline{A}\right) \cdot \overline{ds}$$

Ampere's Circuital Law

Choosing any surface S bounded by the border line L and applying Stokes' theorem to the magnetic field intensity vector $\overline{H}$, we have

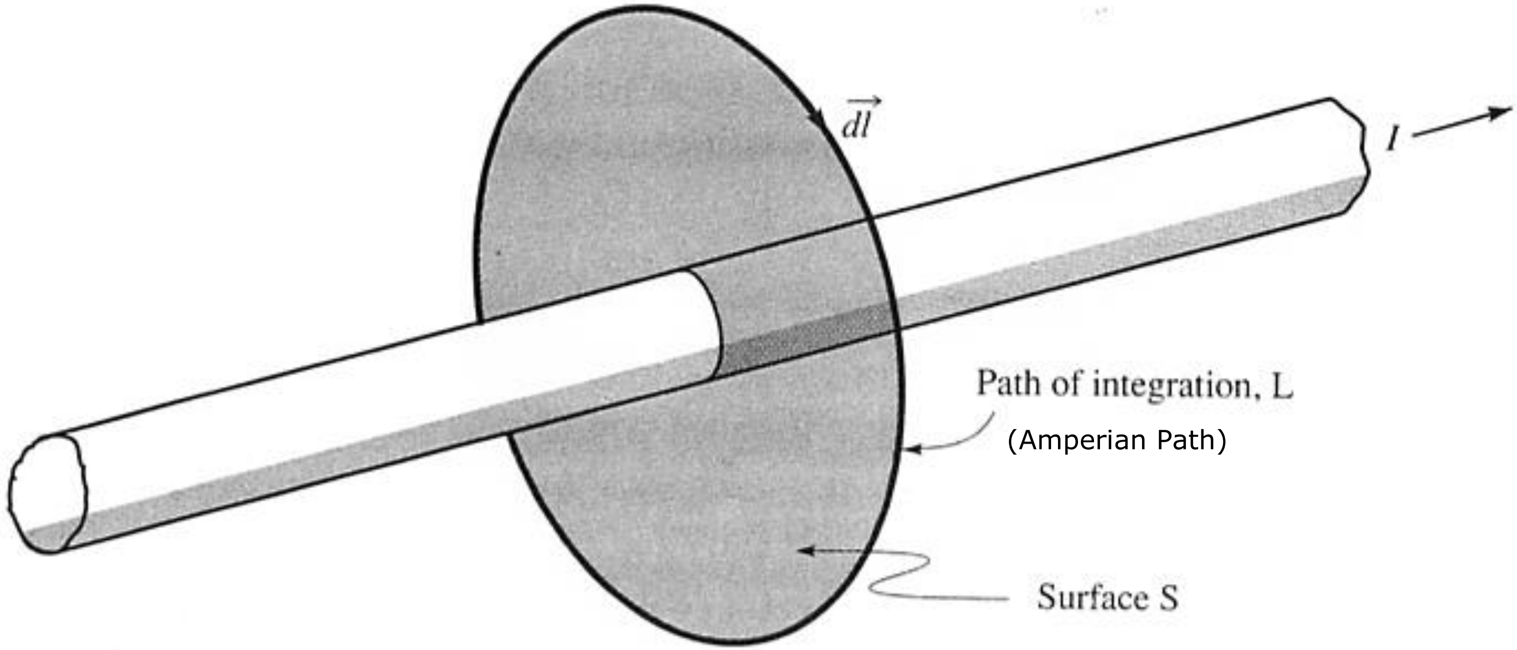
$$\int_S \left(\nabla \times \overline{H}\right)\cdot d\overline{s} = \oint_L \overline{H} \cdot \overline{dl}$$

Substituting the magnetostatic curl equation $\nabla \times \overline{H} = \overline{J}$

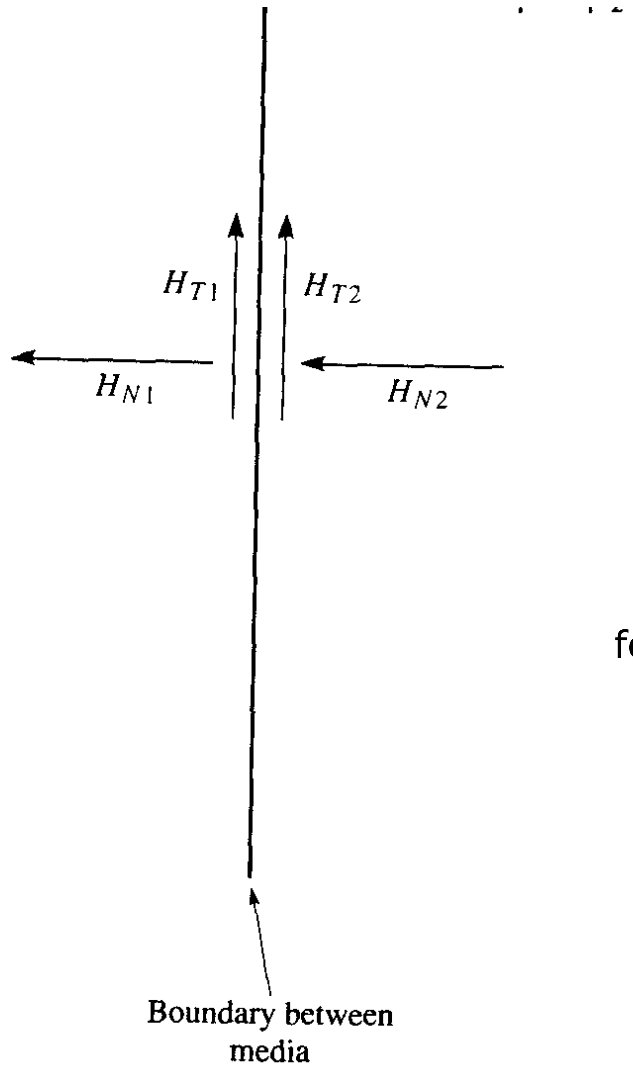
we obtain

$$\overbrace{\int_S \overline{J} \cdot d\overline{s}}^{I_{enc}} = \oint_L \overline{H} \cdot \overline{dl}$$

which is Ampere's Circuital Law. It states that the circulation of $\overline{H}$ around a closed path is equal to the current enclosed by the path.



Ampere’s law is very useful in determining $\overline{H}$ when there is a closed path L around the current I such that the magnitude of $\overline{H}$ is constant over the path.



Boundary conditions are the rules that relate fields on opposite sides of a boundary.

T= tangential components

N= normal components

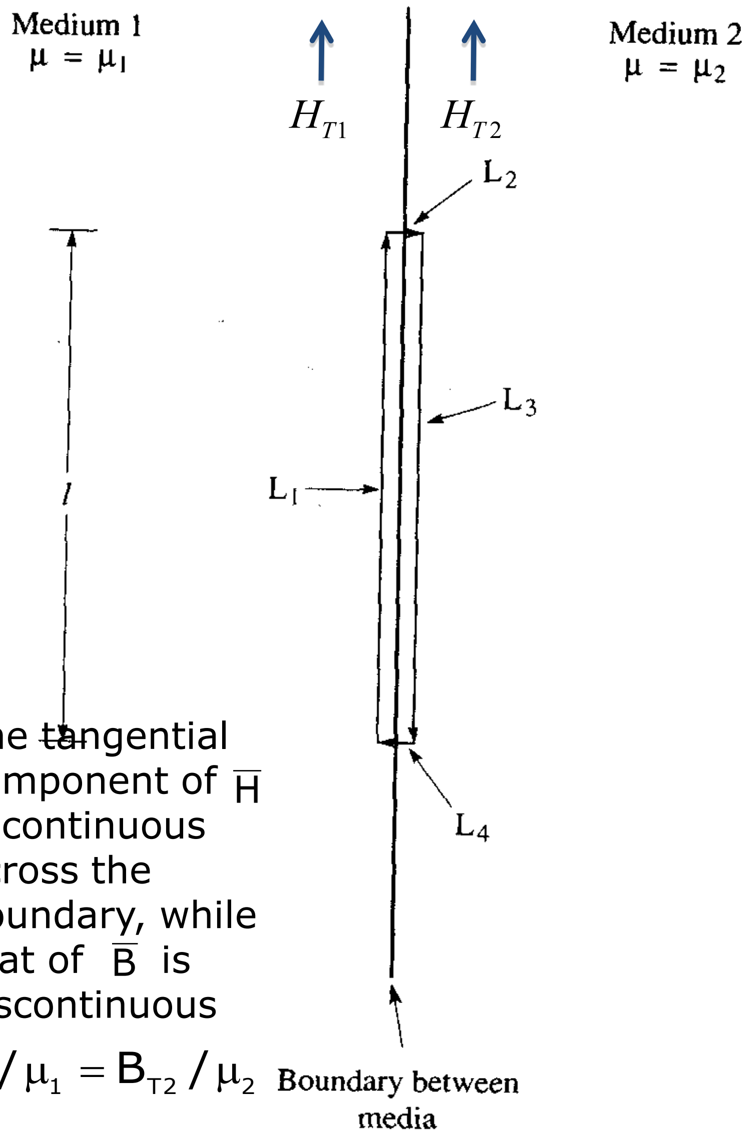
We will make use of Gauss's law for magnetic fields

for B_{N1} and B_{N2} $\rightarrow \int \overline{B} \cdot \overline{ds} = 0$

and Ampere's circuital law

for H_{T1} and H_{T2} $\rightarrow \oint \overline{H} \cdot \overline{dl} = I_{enc}$

Boundary condition on the tangential component of $\overline{H}$



The tangential component of $\overline{H}$ is continuous across the boundary, while that of $\overline{B}$ is discontinuous

$$B_{T1}/\mu_1 = B_{T2}/\mu_2$$

In the limit as $L_2 \to 0$

$L_4 \to 0$ we have

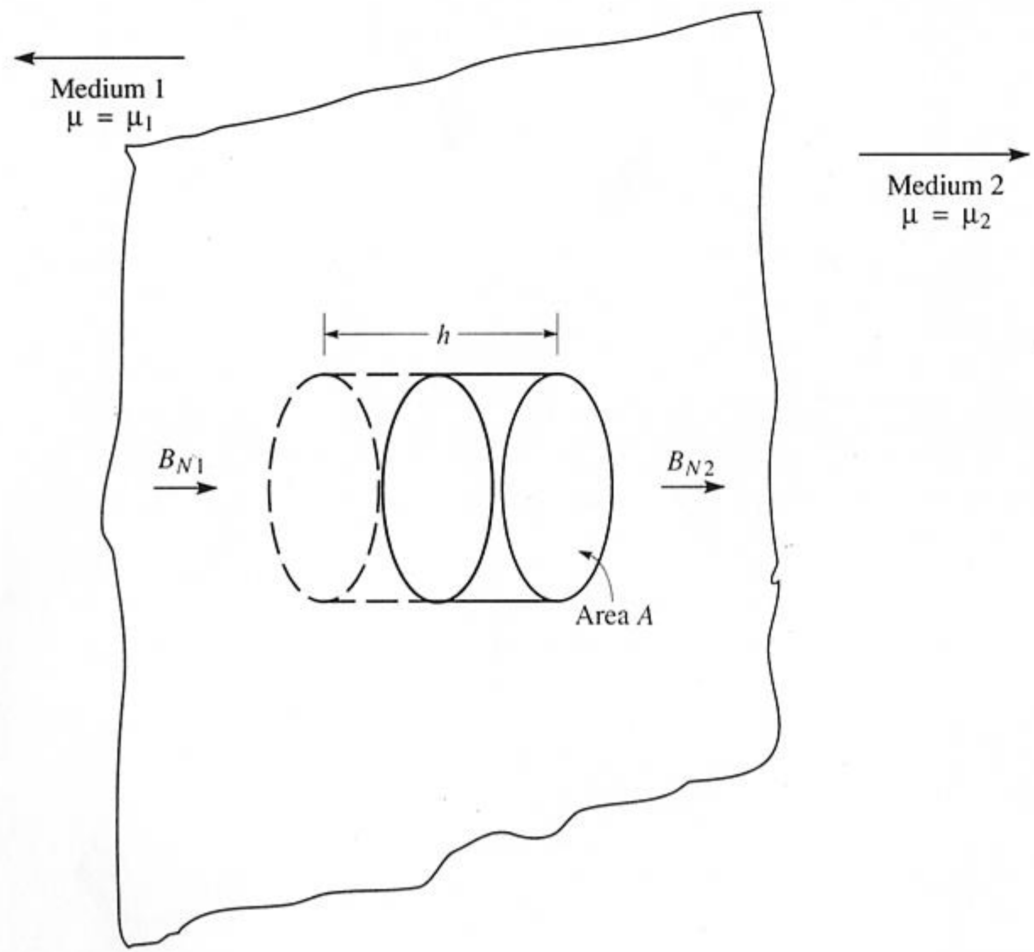
$$\oint_L \overline{H} \cdot \overline{dl} = H_{T1} l - H_{T2} l = I_{enc}$$

where I_{enc} is the current on the boundary surface

(since the integration path in the limit is infinitely narrow).

When the conductivities of both media are finite,

$$I_{enc} = 0 \quad \text{and} \quad H_{T1} = H_{T2}$$



Gaussian surface (cylinder with its plane faces parallel to the boundary)

Applying the divergence theorem to $\overline{B}$ and noting that $\nabla \cdot \overline{B} = 0$ (magnetic field is solenoidal) we have

$$\int_S \overline{B} \cdot \overline{ds} = \int_V \nabla \cdot \overline{B} = 0$$

In the limit as $h \to 0$, the surface integral over the curved surface of the cylinder vanishes. Thus we have

$$B_{N2}\Delta A - B_{N1}\Delta A = 0$$

$$B_{N1} = B_{N2}$$

(the normal component of $\overline{B}$ is continuous at the boundary)

$$\mu_1 H_{N1} = \mu_2 H_{N2}$$

(the normal component of $\overline{H}$ is discontinuous at the boundary)

Future Scope and relevance to industry

- https://pure.tue.nl/ws/portalfiles/portal/90464432/general_formulation_magnetostatic_15_.pdf
- https://lib.dr.iastate.edu/cgi/viewcontent.cgi?referer=https://www.google.co.in/&httpsredir=1&article=12115&context=rtd
- https://www.researchgate.net/publication/3096032_FEM_analysis_of_3D_magnetostatic_fields